Proper Generalized Decomposition for Non Convex Semilinear Problems

A. El Hamidi

Laboratoire LaSIE, Université de La Rochelle. 17042 La Rochelle - France.
aelhamid@univ-lr.fr
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Résumé

Dans ce travail, la méthode PGD est étudiée pour une classe de problèmes semi-linéaires elliptiques non convexes. En utilisant la variété de Nehari associée au problème, on montre que l'on peut récupérer la semi-continuité inférieure faible de l'énergie et ainsi définir une suite PGD convergeant faiblement vers une solution minimisante (sur la variété de Nehari) de type point-selle.

Abstract

In this work, Proper Generalized Decomposition (PGD) is studied for a class of non-convex semilinear elliptic problems. Using the Nehari manifold, we show how to recover the weak semi-continuity property of the associated energy and get a well-defined PGD sequence. The convergence of such a PGD sequence is studied under suitable hypotheses. In particular, the PGD is studied for solutions of saddle-points type.

Mots clefs : Proper Generalized Decomposition, Nehari Manifold, Non convex semilinear problems.

1 Introduction (16 gras)

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with Lipschitz boundary $\partial\Omega$ and $W = H_0^1(\Omega)$ and $p : \mathbb{R} \longrightarrow \mathbb{R}$ be a $C^1(\mathbb{R})$ function satisfying:

$$\exists a_1 \geq 0,\ a_2 \geq 0,\ \forall s \in \mathbb{R}\ :\ |p(s)| \leq a_1 + a_2|s|^\ell,$$

where $\ell < \frac{N+2}{N-2}$ if $N > 2$ and ℓ is unrestricted if $N = 2$. Consider the function $P(s) := \displaystyle\int_0^s p(\tau)d\tau$, then $|P(s)| \leq a_1|s| + a_2|s|^{\ell+1}$, $\forall s \in \mathbb{R}$. Notice that, if $N > 2$, then $\ell + 1 < 2^*$, the Sobolev critical exponent corresponding to W. This makes sense to define $\varphi : W \longrightarrow \mathbb{R}$:

$$\varphi(u) := \int_\Omega P(u(x))\, dx,$$

and hence $\varphi'(u) \cdot w = \int_\Omega p(u(x))\, w(x)\, dx$, for every direction $w \in W$.

Let us introduce the functional defined on W by

$$J(u) := \frac{1}{2} \int_\Omega |\nabla u(x)|^2 dx - \varphi(u),$$

then critical points to J are weak solutions to the semilinear Dirichlet boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta u & = & p(u) & \text{in} \quad \Omega, \\ u & = & 0 & \text{on} \quad \partial\Omega. \end{array} \right. \tag{1}$$

It is clear that the functional J is not necessarily convex, neither bounded above nor bounded below. To overcome this difficulty, we introduce a natural constraint subset containing all critical points of J, more precisely, the Nehari manifold associated to Problem (3) [3, 4]

$$\mathcal{N} := \{u \in W \setminus \{0\} \quad : \quad J'(u) \cdot u = 0\}.$$

We show under further suitable conditions on p that the functional J is bounded and weakly lower semi-continuous on the Nehari manifold $\mathcal{N}$. Moreover, classical variational methods imply that Problem (3) admits a special non trivial solution realizing the infimum [1]

$$\inf_{u \in \mathcal{N}} J(u). \tag{2}$$

2 Proper Generalized Decomposition for minimizing solution

For the convenience of the reader, we recall the notions of *Tensor Banach/Hilbert spaces* and the principle of PGD methods.

2.1 Tensor Banach Spaces and PGD

Consider a finite family of real Banach spaces $(V_k\ , \|\cdot\|_k)_{1\leq k\leq d}$, where $d \geq 2$ is an integer. The algebraic tensor Banach space spanned by the the family $(V_k\ , \|\cdot\|_k)_{1\leq k\leq d}$, denoted by

$$V := {}_a \bigotimes_{k=1}^{d} V_k,$$

is the set of all finite linear combinations of elementary tensors $v = \bigotimes_{k=1}^{d} v_k$, with $v_k \in V_k$. The suffix "a" in "${}_a \bigotimes$" refers to the "algebraic" nature of the tensor product.

We say that $V_{\|\cdot\|}$ is a tensor Banach space if it exists an algebraic tensor space V and a norm $\|\cdot\|$ on V such that $V_{\|\cdot\|}$ is the completion of V with respect to the norm $\|\cdot\|$, *i. e.,*

$$V_{\|\cdot\|} = \overline{{}_a\bigotimes_{k=1}^{d} V_k}^{\|\cdot\|}.$$

In the special case where $V_{\|\cdot\|}$ is a Hilbert space, on says that $V_{\|\cdot\|}$ is a tensor Hilbert space.

In the sequel, the collection of all elementary tensors $v = \bigotimes_{k=1}^{d} v_k$, with $v_k \in V_k$, will be denoted by $\mathcal{R}_1(V)$, thus the space spanned by $\mathcal{R}_1(V)$ is V and consequently $\mathrm{span}\,(\mathcal{R}_1(V))$ is a dense subset of $V_{\|\cdot\|}$.

Let us return to Problem (2). Notice that if $\Omega = \prod_{i=1}^{N} \Omega_i$, where $(\Omega_i)_{1\le i\le N}$ is a family of bounded open real intervals, then W is a tensor Hilbert space and we then we can construct a PGD sequence as the following:

$$(M) \quad \left\{\begin{array}{ll} (i) & Initialization : u_0 = 0 \; in \; W. \\ (ii) & Descent\, direction: \; z_m := \underset{z\in(-u_{m-1}+\mathcal{N})\cap\mathcal{R}_1(W)}{\mathrm{argmin}} J(u_{m-1}+z) \\ (iii) & Update: \; u_m := u_{m-1} + z_m, \; \forall\, m \ge 1, \end{array}\right.$$

where $\mathcal{R}_1(W)$ is the set of all elementary tensors in W [2].

We show, under suitable conditions on the function p, that we can recover the weak convergence of the PGD sequence $(u_m)_{m\in\mathbb{N}}$ defined by the scheme (M) toward the solution of Problem (2).

2.2 Example of super-linear subcritical elliptic equation

Consider the boundary value problem

$$\left\{\begin{array}{rcll} -\Delta u &=& u^3 & \text{in} \quad \Omega, \\ u &=& 0 & \text{on} \quad \partial\Omega. \end{array}\right. \tag{3}$$

where $\Omega = \prod_{i=1}^{N} \Omega_i$, $N \ge 3$. In this case, the Nehari manifold

$$\mathcal{N} = \{\pm t(u)\, u \;:\; u \in W \setminus \{0\}\},$$

where

$$t(u) = \frac{\|\nabla u\|_2^2}{\|u\|_4^4} \quad \text{and} \quad \mathrm{J}(\pm t(u)u) = \frac{1}{4}\left(\frac{\|\nabla u\|_2}{\|u\|_4}\right)^4.$$

Let us define $\Phi : W \setminus \{0\} \longrightarrow W \setminus \{0\}$ by $\Phi(u) := t(u)u$ and $\mathcal{J} := J \circ \Phi$. It is of interest to remark that for given $u \notin \mathcal{R}_1(W)$, we have

$$\inf_{z \in (-u+\mathcal{N}) \cap \mathcal{R}_1(W)} J(u+z) \Longleftrightarrow \inf_{z \in \mathcal{R}_1(W)} \mathcal{J}(u+z),$$

which simplifies substantially the minimization problem.

We can verify easily that the functional $\mathcal{J}$ is bounded below and weakly lower semi-continous. Hence, the minimization problem (ii) of the scheme (M) admits a solution and therefore the PGD sequence is well-defined. On the other hand, we show that if there is an integer m such that

$$\inf_{z \in \mathcal{R}_1(W)} \mathcal{J}(u_m + z) = 0_W$$

then $\Phi(u_m)$ is a solution of (2). We end this work by some observations about the weak convergence of the sequence $(\Phi(u_m))_m$ when the sequence $(\mathcal{J}(u_m))_m$ is strictly decreasing.

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